Quantitative and Mathematical Methods
Euro-American Campus · Sciences Po · Reims

Functions

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Level 1 Groups
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- **Definition:**

  The real-valued function \( f : x \rightarrow y \quad x, y \in \mathbb{R} \) is a rule that assigns *one* real number \( y \in \mathbb{R}^1 \) to each real number \( x \in \mathbb{R}^1 \).  

\( \mathbb{R}^1 \) denotes the set of all real numbers from \(-\infty\) to \( +\infty \) on the real number line.
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- **Domain/Range:**

  Given \( f : x \in X \rightarrow y \in Y \), the set \( X \) denotes the *domain* of the function and the set \( Y \) denotes its *range* (co-domain).
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- **Euclidean space:**
  A function with \( n \) variables exists in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) where each \( n \)-axis goes from \(-\infty\) to \(+\infty\).
Functions: Set notation

- Let $X$ be a set:

  - $x \in X$: the element $x$ belongs to the set $X$.
  - $x \notin X$: the element $x$ does not belong to the set $X$.
  - $X = \{\emptyset\}$: empty set.
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- **Important sets:**

  $\mathbb{N}$: natural numbers
  
  $\mathbb{R}$: real numbers
  
  $\mathbb{Z}$: integers
  
  $\mathbb{Q} = \{n/d : (n, d) \in \mathbb{Z} \text{ and } d \neq 0\}$: rational numbers
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  - $\mathbb{N}$: natural numbers
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- **Interval notation:**

  $[a, b] = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}$: closed interval
  $(a, b) = \{x : x \in \mathbb{R} \text{ and } a < x < b\}$: open interval
Functions: Relation to sets

- Let $X$ and $Y$ be sets:
  - $X = Y$: the sets $X$ and $Y$ are equal.
  - $X \subseteq Y$: $X$ is a subset of $Y$.
  - $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$: intersection
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- **Cartesian product:**

  The Cartesian product \( X \times Y \) of two sets \( X \) and \( Y \) is the set of all ordered pairs \((x, y)\) with \( x, y \in \mathbb{R} \).

  **Ex.** Let \( X = \{1, 2\} \) and \( Y = \{2, 4\} \).
  
  Then \( X \times Y = (1, 2), (1, 4), (2, 2), (2, 4) \).
Functions: Relation to sets

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  $X = Y$: the sets $X$ and $Y$ are equal.
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  The Cartesian product $X \times Y$ of two sets $X$ and $Y$ is the set of all ordered pairs $(x, y)$ with $x, y \in \mathbb{R}$.

  Ex. Let $X = \{1, 2\}$ and $Y = \{2, 4\}$.
  Then $X \times Y = (1, 2), (1, 4), (2, 2), (2, 4)$.

• In more general terms:

  Let $X$ and $Y$ be sets. The relation $R$ from $X$ to $Y$ is a subset of $X \times Y$ and is written $xRy$ if $(x, y) \in R$. 
Functions: Terminology

- **Functions as mappings of sets:**

  The basic point of a function is to provide a ‘rule of correspondence’ between two sets: the element \( x \in X \) is the input of the function, which produces the output \( y \in Y \).
Functions: Terminology

- **Functions as mappings of sets:**
  
  The basic point of a function is to provide a ‘rule of correspondence’ between two sets: the element $x \in X$ is the **input** of the function, which produces the **output** $y \in Y$.

- **Functions as dependence:**
  
  Using the function $f : x \rightarrow y$, we might choose to express the relationship between two terms $x$ and $y$ as a **dependence** of $y$ upon $x$. We might later say that $x$ **predicts** $y$. 
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  The basic point of a function is to provide a ‘rule of correspondence’ between two sets: the element \( x \in X \) is the **input** of the function, which produces the **output** \( y \in Y \).

- **Functions as dependence:**
  Using the function \( f : x \rightarrow y \), we might choose to express the relationship between two terms \( x \) and \( y \) as a *dependence* of \( y \) upon \( x \). We might later say that \( x \) **predicts** \( y \).

- **Back to domains:**
  Let \( X \) and \( Y \) be sets. A function \( f : X \rightarrow Y \) is a relation from its **domain** \( X \) to its **codomain** \( Y \). The set \( Y = f(X) \) can be written as \( \{ f(x) : x \in X \} \) and is the **image** (or **range**) of \( f \).
Functions: Examples
Functions: Example

- Model of market commodity:

  **Demand function:** \( D(x) = p \) where \( p \) is the price at which each unit of commodity \( x \) sells.

  **Supply function:** \( S(x) = p \) where \( p \) is the price at which units of \( x \) are effectively sold.

  How do these functions behave in economic view?
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  How do these functions behave in economic view?

- Model of market profit:

  **Revenue function:** \( R(x) = x \cdot p(x) \), which stands for (number of units sold) \( \times \) (price per unit).
  
  **Cost function:** \( C(x) \)

  **Profit function:**

\[
P(x) = R(x) - C(x) = x \cdot p(x) - C(x)
\]
Since negative numbers do not have real square roots, \( g(t) \) can be evaluated only when \( t /H110022 /H113500 \), so the ... or if \( a \cdot 0 \) and \( b \cdot 0 \).
Functions: Applied example 1

- Assume the following commodity pricing situation:

\[ p(x) = -0.27x + 51C(x) = 2.23x^2 + 3.5x + 85 \]

Find \( R(x) \) and \( P(x) \).
Functions: Applied example 1

- Assume the following commodity pricing situation:

\[ p(x) = -0.27x + 51C(x) = 2.23x^2 + 3.5x + 85 \]

Find \( R(x) \) and \( P(x) \).

- Solution 1:

\[ R(x) = x \cdot p(x) = -0.27x^2 + 51x \]
\[ P(x) = R(x) - C(x) = (-0.27x^2 + 51x) - (2.23x^2 + 3.5x + 85) \]
\[ \Rightarrow P(x) = -2.5x^2 + 47.5x - 85 \]
Functions: Applied example 2

- Following up on Example 1:

\[ P(x) = -2.5x^2 + 47.5x - 85 \text{ dollars} \]

At what values of \( x \) is production profitable?
Functions: Applied example 2

- **Following up on Example 1:**

  \[ P(x) = -2.5x^2 + 47.5x - 85 \text{ dollars} \]

  At what values of \( x \) is production profitable?

- **Solution 2:**

  \[ P(x) = -2.5x^2 + 47.5x - 85 \]
  \[ = -2.5(x^2 - 19x + 34) \]
  \[ = -2.5(x - 2)(x - 17) \]

  Production is profitable if \( P(x) > 0 \).
  \( P(x) > 0 \) only if \( (x - 2)(x - 17) \) is negative, i.e. when \( x - 2 > 0 \) and \( x - 17 < 0 \).
  Production is profitable if \( 2 < x < 17 \).
Functions: Applied example 3

- Suppose the following:
  Fixed cost function:

  \[ C(q) = q^3 - 30q^2 + 500q + 200 \]

  Compute the production cost of 10 units.
Functions: Applied example 3

- **Suppose the following:**
  **Fixed cost function:**

  \[ C(q) = q^3 - 30q^2 + 500q + 200 \]

  Compute the production cost of 10 units.

- **Solution 3:**
  \[ C(10) = (10)^3 - 30(10)^2 + 500(10) + 200 = 3,200. \]
Assume the following fixed cost function:

\[ C(q) = q^3 - 30q^2 + 500q + 200 \]

\[ C(10) = 3,200. \] Compute the production cost of the 10th unit.
Functions: Applied example 4

• Assume the following fixed cost function:

\[ C(q) = q^3 - 30q^2 + 500q + 200 \]

\[ C(10) = 3,200. \text{ Compute the production cost of the } 10\text{th unit.} \]

• Solution 4:

\[ C(10) = 3,200 \]
\[ C(9) = (9)^3 - 30(9)^2 + 500(9) + 200 = 2,999 \]

Marginal cost of 10th unit: \( C(10) - C(9) = 201. \)